Mountain pass and symmetric mountain pass approaches for nonlinear scalar field equations (Variational Problems and Related Topics)

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1. Introduction

This note is based on my joint work [**HIT**] with J. Hirano and N. Ikoma and we study the existence of radially symmetric solutions of the following nonlinear scalar field equations:

$$-\Delta u = g(u) \quad \text{in } \mathbf{R}^N, \tag{1.1}$$

$$u \in H^1(\mathbf{R}^N). \tag{1.2}$$

Here $N \geq 2$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function. This type of problem appears in many models in mathematical physics etc. and almost necessary and sufficient conditions for the existence of non-trivial solutions are obtained by Berestycki-Lions [**BL1**, **BL2**] for $N \geq 3$ and Berestycki-Gallouët-Kavian [**BGK**] for $N = 2$. See also Strauss [**Sw**] and Coleman-Glaser-Martin [**CGM**] for earlier works.

In [**BL1**, **BL2**, **BGK**] they assume

(g0) $g(\xi) \in C(\mathbf{R}, \mathbf{R})$ and $g(\xi)$ is odd.

(g1) For $N \geq 3$, $\limsup_{\xi\to\infty} \frac{g(\xi)}{\xi^{\frac{N+2}{N-2}}} \leq 0$. For $N = 2$, $\limsup_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} \leq 0$ for any $\alpha > 0$.

(g2) For $N \geq 3$

$$-\infty < \liminf_{\xi\to 0} \frac{g(\xi)}{\xi} \leq \limsup_{\xi\to 0} \frac{g(\xi)}{\xi} < 0. \tag{1.3}$$

For $N = 2$

$$-\infty < \lim_{\xi\to 0} \frac{g(\xi)}{\xi} < 0. \tag{1.4}$$

(g3) There exists a $\zeta_0 > 0$ such that $G(\zeta_0) > 0$, where $G(\xi) = \int_0^\xi g(\tau)\, d\tau$.

Under the above conditions, they show the existence of *a positive solution* and *infinitely many (possibly sign changing) radially symmetric solutions.*

(g0)–(g3) are natural conditions for the existence of solutions. However we can see a difference between cases $N \geq 3$ and $N = 2$ in the condition (g2). We remark that when $N = 2$, the existence of a limit $\lim_{\xi\to 0} \frac{g(\xi)}{\xi} \in (-\infty, 0)$ is used essentially to show the Palais-Smale compactness condition for the corresponding functional under suitable constraint ([**BGK**]).

The aim of this paper is to extend the result of [**BGK**] slightly and we prove the existence of positive solution and infinitely many radially symmetric solutions under the conditions (g0), (g1), (g3) and (1.3) (not (1.4)).

We also remark that in [**BL1**, **BL2**, **BGK**], they constructed solutions of (1.1)–(1.2) through constraint problems in the space of radially symmetric functions:

$$\text{find critical points of } \left\{\int_{\mathbf{R}^N} |\nabla u|^2\, dx\,;\ \int_{\mathbf{R}^N} G(u)\, dx = 1\right\}\ (N \geq 3), \tag{1.5}$$

or

$$\text{find critical points of } \left\{\int_{\mathbf{R}^2} |\nabla u|^2\, dx\,;\ \int_{\mathbf{R}^2} G(u)\, dx = 0,\ \int_{\mathbf{R}^2} u^2\, dx = 1\right\}\ (N = 2). \tag{1.6}$$

In fact, if $v(x)$ is a critical point of (1.5) or (1.6), then for a suitable $\lambda > 0$, $u(x) = v(x/\lambda)$ is a solution of (1.1)–(1.2). On the other hand, solutions of (1.1)–(1.2) are also characterized as critical points of the functional $I(u) \in C^1(H^1_r(\mathbf{R}^N), \mathbf{R})$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \int_{\mathbf{R}^N} G(u)\, dx.$$

Here we denote by $H^1_r(\mathbf{R}^N)$ the space of radially symmetric H^1-functions defined on $\mathbf{R}^N$. It is natural to ask whether it is possible to find critical points through the unconstraint functional $I(u)$.

Our second aim is to give another proof of the results of [**BL1**, **BL2**, **BGK**] using mountain pass and symmetric mountain pass arguments to $I(u)$.

Now we can state our main result.

Theorem 1.1. *Assume $N \geq 2$ and (g0), (g1), (g3) and*

(g2') $-\infty < \liminf_{\xi\to 0} \dfrac{g(\xi)}{\xi} \leq \limsup_{\xi\to 0} \dfrac{g(\xi)}{\xi} < 0.$

Then (1.1)–(1.2) has a positive least energy solution and infinitely many (possibly sign changing) radially symmetric solutions, which are characterized by the mountain pass and symmetric mountain pass minimax arguments in $H^1_r(\mathbf{R}^N)$ (see (2.1)–(2.2), (3.11) below for a mountain pass minimax values).

We will take mountain pass and symmetric mountain pass approaches to prove Theorem 1.1. Since $I(u)$ is an even functional with a mountain pass geometry and it is possible to define a mountain pass minimax value b_{mp} and symmetric mountain pass values b_n ($n \in \mathbf{N}$) for $I(u)$. By the Ekeland's principle, we can find a Palais-Smale sequence $(u_j)_{j=1}^{\infty} \subset H_r^1(\mathbf{R}^N)$ at levels b_{mp} and b_n, that is, $(u_j)_{j=1}^{\infty}$ satisfies

$$I(u_j) \to b_{mp} \text{ (or } b_n\text{)}, \tag{1.7}$$

$$I'(u_j) \to 0 \text{ strongly in } (H_r^1(\mathbf{R}^N))^*. \tag{1.8}$$

However one of the difficulty is the lack of the Palais-Smale compactness condition and it seems difficult to show the existence of strongly convergent subsequence merely under the conditions (1.7)–(1.8). A key of our argument is to find a Palais-Smale sequence with an extra property related to Pohozaev identity.

In the following sections, we give an outline of our approach to give a proof of our Theorem 1.1. For the sake of simplicity, we just deal with the existence of a positive solution, which is corresponding to the moutain pass theorem. For a proof for symmetric moutain pass minimax values we refer to [**HIT**].

2. Mountain pass minimax value

Modifying the nonlinearity $g(\xi)$ as in [**BL1**] if necessary, we may assume that $g(\xi)$ satisfies (g0), (g2'), (g3) and

(g1') when $N \geq 3$, $\displaystyle\lim_{\xi\to\infty} \frac{g(\xi)}{|\xi|^{\frac{N+2}{N-2}}} = 0$.

when $N = 2$, $\displaystyle\lim_{\xi\to\infty} \frac{g(\xi)}{e^{\alpha\xi^2}} = 0$ for any $\alpha > 0$.

We remark that under (g0), (g1'), (g2'), (g3)

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \int_{\mathbf{R}^N} G(u)\, dx : H_r^1(\mathbf{R}^N) \to \mathbf{R}$$

is of class C^1 and has a mountain pass geometry. That is,

(i) $I(0) = 0$.

(ii) There exist $r_0 > 0$ and $\rho_0 > 0$ such that

$$I(u) \geq \rho_0 \quad \text{for } \|u\|_{H^1} = r_0.$$

(iii) There exists an $e_0(x) \in H_r^1(\mathbf{R}^N)$ such that $\|e_0\|_{H^1} > r_0$ and

$$I(e_0) < 0.$$

Here we use notation: $\|u\|_{H^1} = (\int_{\mathbf{R}^N} |\nabla u|^2 + |u|^2\, dx)^{1/2}$.
We can define the mountain pass minimax value b_{mp} by

$$b_{mp} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{2.1}$$

$$\Gamma = \{\gamma \in C([0,1], H_r^1(\mathbf{R}^N));\ \gamma(0) = 0,\ \gamma(1) = e_0\}. \tag{2.2}$$

As we stated in the Introduction, it is difficult to check the Palais-Smale condition for $I(u)$ and it is a main difficulty in proving Theorem 1.1.

3. Our approach

To show b_{mp} is a critical value of $I(u)$, first we recall that $\int_{\mathbf{R}^N} |\nabla u|^2\, dx$ and $\int_{\mathbf{R}^N} G(u)\, dx$ have the following scaling property: for $u_\theta(x) = u(e^{-\theta} x)$

$$\int_{\mathbf{R}^N} |\nabla u_\theta|^2\, dx = e^{(N-2)\theta} \int_{\mathbf{R}^N} |\nabla u|^2\, dx,$$

$$\int_{\mathbf{R}^N} G(u_\theta)\, dx = e^{N\theta} \int_{\mathbf{R}^N} G(u)\, dx.$$

We introduce an auxiliary functional $\widetilde{I}(\theta, u) \in C^1(\mathbf{R} \times H_r^1(\mathbf{R}^N), \mathbf{R})$ by

$$\widetilde{I}(\theta, u) = \frac{1}{2} e^{(N-2)\theta} \int_{\mathbf{R}^N} |\nabla u|^2\, dx - e^{N\theta} \int_{\mathbf{R}^N} G(u)\, dx$$

which has the following properties:

$$\widetilde{I}(0, u) = I(u), \tag{3.1}$$

$$\widetilde{I}(\theta, u) = I(u_\theta) \quad \text{for all } \theta \in \mathbf{R} \text{ and } u \in H_r^1(\mathbf{R}^N). \tag{3.2}$$

We equip a standard product norm $\|(\theta, u)\|_{\mathbf{R} \times H^1} = (|\theta|^2 + \|u\|_{H^1}^2)^{1/2}$ to $\mathbf{R} \times H_r^1(\mathbf{R}^N)$.

Remark 3.1. We remark that this type of auxiliary functionals was first used in Jeanjean [**J1**] for a nonlinear eigenvalue problem. It should be compared with monotonicity method due to Struwe [**Sm**] and Jeanjean [**J2**]. We expect that this type of auxiliary functionals can be applied to other problems.

We define minimax values $\tilde{b}_{mp}$ for $\widetilde{I}(\theta, u)$ by

$$\tilde{b}_{mp} = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \widetilde{I}(\tilde{\gamma}(\sigma)),$$

$$\tilde{\Gamma} = \{\tilde{\gamma}(t) \in C([0,1], \mathbf{R} \times H_r^1(\mathbf{R}^N));\ \tilde{\gamma}(0) = (0,0),\ \tilde{\gamma}(1) = (0, e_0)\}.$$

Then we have

Lemma 3.2. $\tilde{b}_{mp} = b_{mp}$.

Proof. For any $\gamma \in \Gamma$ we can see that $(0, \gamma(t)) \in \tilde{\Gamma}$ and we may regard $\Gamma \subset \tilde{\Gamma}$. Thus by the definitions of b_{mp}, $\tilde{b}_{mp}$ and (3.1), we have $\tilde{b}_{mp} \leq b_{mp}$. Next for any given $\tilde{\gamma}(t) = (\theta(t), \eta(t)) \in \tilde{\Gamma}_n$, we set $\gamma(t) = \eta(t)(e^{-\theta(t)}x)$. We can verify that $\gamma(t) \in \Gamma$ and by (3.2) $I(\gamma(t)) = \widetilde{I}(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Thus we also have $\tilde{b}_n \geq b_n$. ∎

As a virtue of $\widetilde{I}(\theta, u)$ we can find a Palais-Smale sequence (θ_j, u_j) in the augmented space $\mathbf{R} \times H_r^1(\mathbf{R}^N)$ with an additional property (3.6) below. Namely we have

Proposition 3.3. *For any $n \in \mathbf{N}$ there exists a sequence $(\theta_j, u_j)_{j=1}^{\infty} \subset \mathbf{R} \times H_r^1(\mathbf{R}^N)$ such that*

(i) $\theta_j \to 0$. (3.3)

(ii) $\widetilde{I}(\theta_j, u_j) \to b_{mp} \, (= \tilde{b}_{mp})$. (3.4)

(iii) $\widetilde{I}'(\theta_j, u_j) \to 0$ *strongly in* $(H_r^1(\mathbf{R}^N))^*$. (3.5)

(iv) $\dfrac{\partial}{\partial\theta}\widetilde{I}(\theta_j, u_j) \to 0$. (3.6)

Proof. We note that for any $\varepsilon > 0$ there exists a path $\gamma(t) \in \Gamma \subset \tilde{\Gamma}$ such that

$$\max_{t \in [0,1]} \widetilde{I}(0, \gamma(t)) \leq \tilde{b}_{mp} + \varepsilon.$$

Applying Ekeland's principle in the product space $\mathbf{R} \times H_r^1(\mathbf{R}^N)$, we can show Proposition 3.3 in a standard way. ∎

Next we study boundedness and compactness of a sequence (θ_j, u_j) given in Proposition 3.3. First we observe that (3.4) and (3.6) imply the following

$$\frac{1}{2}e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 - e^{N\theta_j}\int_{\mathbf{R}^N} G(u_j)\,dx \to b_{mp}, \tag{3.7}$$

$$\frac{N-2}{2}e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 - Ne^{N\theta_j}\int_{\mathbf{R}^N} G(u_j)\,dx \to 0 \quad \text{as } j \to \infty. \tag{3.8}$$

Remark 3.4. We recall that if $u(x)$ is a critical point of $I(u)$, then $u(x)$ satisfies

$$P(u) = 0, \quad \text{where} \quad P(u) = \frac{N-2}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - N\int_{\mathbf{R}^N} G(u)\,dx.$$

The above equality is called *Pohozaev identity.* We remark that since $\theta_j \to 0$, (3.8) gives a property related to the Pohozaev identity.

It follows from (3.7)-(3.8) that

$$\|\nabla u_j\|_2^2 \to Nb_n, \tag{3.9}$$

$$\int_{\mathbf{R}^N} G(u_j)\,dx \to \frac{N-2}{2}b_n.$$

We can show

Proposition 3.5. *Let (θ_j, u_j) be a sequence satisfying* (3.3)–(3.6). *Then we have*
(i) (u_j) is bounded in $H_r^1(\mathbf{R}^N)$.
(ii) (u_j) has a strongly convergent subsequence in $H_r^1(\mathbf{R}^N)$.

Proof. (i) When $N \geq 3$, we can prove Proposition 3.5 in a direct way. Indeed, by (g0), (g1'), (g2') there exists a small $m_0 > 0$ such that

$$m_0\xi^2 + g(\xi)\xi \leq C|\xi|^{\frac{N+2}{N-2}} \quad \text{for all } \xi \in \mathbf{R}.$$

It follows from $\varepsilon_j = \|\widetilde{I}'(\theta_j, u_j)\|_{(H_r^1(\mathbf{R}^N))^*} \to 0$ that $|\widetilde{I}'(\theta_j, u_j)u_j| \leq \varepsilon_j\|u_j\|_{H^1}$. Thus

$$\begin{aligned} e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + m_0 e^{N\theta_j}\|u_j\|_2^2 &\leq e^{N\theta_j}\int_{\mathbf{R}^N} m_0 u_j^2 + g(u_j)u_j\,dx + \varepsilon_j\|u_j\|_{H^1} \\ &\leq Ce^{N\theta_j}\|u_j\|_{2N/(N-2)}^{2N/(N-2)} + \varepsilon_j\|u_j\|_{H^1}. \end{aligned} \tag{3.10}$$

Since $\|\nabla u_j\|_2$ is bounded by (3.9), we can observe that $\|u_j\|_{2N/(N-2)}$ is also bounded. Thus (3.10) implies boundedness of $\|u_j\|_2$, that is, (u_j) is bounded in $H_r^1(\mathbf{R}^N)$.

To prove (i) when $N = 2$, we need to use a blow-up argument. We refer to [**HIT**] (see also Jeanjean and Tanaka [**JT2**]).

(ii) By the boundedness of $(u_j)_{j=1}^\infty$ in $H_r^1(\mathbf{R}^N)$, we can extract a weakly convergent subsequence — still we denote by j — such that $u_j \rightharpoonup u_0$. We can easily see that $(0, u_0)$ is a critical point of $\widetilde{I}(\theta, u)$. The weak upper semi-continuity of

$$u \mapsto \int_{\mathbf{R}^N} m_0 u^2 + g(u)u\,dx;\ H_r^1(\mathbf{R}^N) \to \mathbf{R},$$

which follows from Fatou's lemma, implies $u_j \to u_0$ strongly in $H_r^1(\mathbf{R}^N)$. ∎

End of the proof of Theorem 1.1. Let (θ_j, u_j) be a sequence obtained in Proposition 3.3. By Proposition 3.5, we may assume $u_j \to u_0$ strongly in $H_r^1(\mathbf{R}^N)$. Then u_0 satisfies

$$\widetilde{I}(0, u_0) = b_{mp} \quad \text{and} \quad \widetilde{I}'(0, u_0) = 0,$$

that is nothing but

$$I(u_0) = b_{mp} \quad \text{and} \quad I'(u_0) = 0, \tag{3.11}$$

Thus b_{mp} is a critical value of $I(u)$.

As to the positivity of a critical point corresponding to b_{mp} and the fact that it has least energy among all non-trivial solutions, we refer to [**HIT**]. ∎

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